Solution

1. At equilibrium the pressure p inside the vessel must be equal to the room pressure p_0 plus the pressure induced by the weight of the movable base: $p = p_0 + \dfrac{mg}{\pi r^2}$. This is true before and after irradiation. Initially the gas temperature is room temperature. Owing to the state equation of perfect gases, the initial gas volume V_1 is $V_1 = \dfrac{nRT_0}{p}$ (where R is the gas constant) and therefore the height h_1 of the cylinder which is occupied by the gas is $h_1 = \dfrac{V_1}{\pi r^2} = \dfrac{nRT_0}{p_0 \pi r^2 + mg}$. After irradiation, this height becomes $h_2 = h_1 + \Delta s$, and therefore the new temperature is
$T_2 = T_0\left(1 + \dfrac{\Delta s}{h_1}\right) = T_0 + \dfrac{\Delta s(p_0 \pi r^2 + mg)}{nR}$.
Numerical values: p = 102.32 kPa; T_2 = 322 K = 49°C

2. The mechanical work made by the gas against the plate weight is $mg\Delta s$ and against the room pressure is $p_0 \pi r^2 \Delta s$, therefore the total work is $L = (mg + p_0 \pi r^2)\Delta s$ = 24.1 J

3. The internal energy, owing to the temperature variation, varies by an amount $\Delta U = nc_V(T_2 - T_0)$.
The heat introduced into the system during the irradiation time Δt is
$Q = \Delta U + L = nc_V \dfrac{T_0 \Delta s}{h_1} + (mg + p_0 \pi r^2)\Delta s = \Delta s\left(p_0 \pi r^2 + mg\right)\left(\dfrac{c_V}{R} + 1\right)$. This heat comes exclusively from the absorption of optical radiation and coincides therefore with the absorbed optical energy, Q = 84 J.

The same result can also be obtained by considering an isobaric transformation and remembering the relationship between molecular heats:
$$Q = nc_p(T_2 - T_0) = n(c_V + R)\left[\frac{\Delta s(p_0 \pi r^2 + mg)}{nR}\right] = \Delta s(p_0 \pi r^2 + mg)\left(\frac{c_V}{R} + 1\right)$$

4. Since the laser emits a constant power, the absorbed optical power is $W = \dfrac{Q}{\Delta t} = \left(\dfrac{c_V}{R} + 1\right)\dfrac{\Delta s}{\Delta t}(p_0 \pi r^2 + mg)$ = 8.4 W. The energy of each photon is hc/λ, and thus the number of photons absorbed per unit time is $\dfrac{W\lambda}{hc}$ = $2.2 \cdot 10^{19}$ s^{-1}

5. The potential energy change is equal to the mechanical work made against the plate weight, therefore the efficiency η of the energy transformation is

$$\frac{mg\Delta s}{Q} = \frac{1}{\left(1+\frac{p_0\pi\, r^2}{mg}\right)\left(1+\frac{c_V}{R}\right)} = 2.8\cdot 10^{-3} \approx 0.3\%$$

6. When the cylinder is rotated and its axis becomes horizontal, we have an adiabatic transformation where the pressure changes from p to p_0, and the temperature changes therefore to a new value T_3. The equation of the adiabatic transformation $pV^{\gamma} = \text{constant}$ may now be written in the form $T_3 = T_2\left(\frac{p_0}{p}\right)^{\frac{\gamma-1}{\gamma}}$, where $\gamma = \frac{c_\mathrm{p}}{c_\mathrm{V}} = \frac{c_\mathrm{V}+R}{c_\mathrm{V}} = 1+\frac{R}{c_\mathrm{V}} = 1.399$. Finally T_3 = 321 K = 48°C

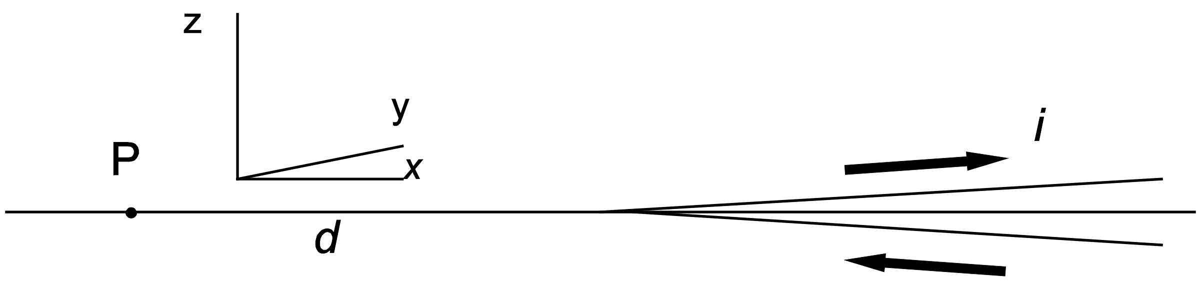
Grading guidelines

1.	0.5	Understanding the relationship between inner and outer pressure
	0.7	Proper use of the plate displacement
	0.2+0.2	Correct results for final pressure
	0.2+0.2	Correct results for final temperature
2.	0.6	Understanding that the work is made both against plate weight and against atmospheric pressure
	0.2+0.2	Correct results for work
3.	1	Correct approach
	0.5	Correct equation for heat
	0.3	Understanding that the absorbed optical energy equals heat
	0.2	Correct numerical result for optical energy
4.	0.2+0.2	Correct results for optical power
	0.5	Einstein's equation
	0.3+0.3	Correct results for number of photons
5.	0.6	Computation of the change in potential energy
	0.2+0.2	Correct results for efficiency
6.	0.8	Understanding that the pressure returns to room value
	0.4	Understanding that there is an adiabatic transformation
	0.4	Equation of adiabatic transformation
	0.5	Derivation of γ from the relationship between specific heats
	0.2+0.2	Correct results for temperature

For "correct results" two possible marks are given: the first one is for the analytical equation and the second one for the numerical value.
For the numerical values a full score cannot be given if the number of digits is incorrect (more than one digit more or less than those given in the solution) or if the units are incorrect or missing.
No bonus can be given for taking into account the gas weight

Solution

1. The contribution to **B** given by each leg of the "V" has the same direction as that of a corresponding infinite wire and therefore - if the current proceeds as indicated by the arrow - the magnetic field is orthogonal to the wire plane taken as the *x*-*y* plane. If we use a right-handed reference frame as indicated in the figure, **B**(P) is along the positive *z* axis.



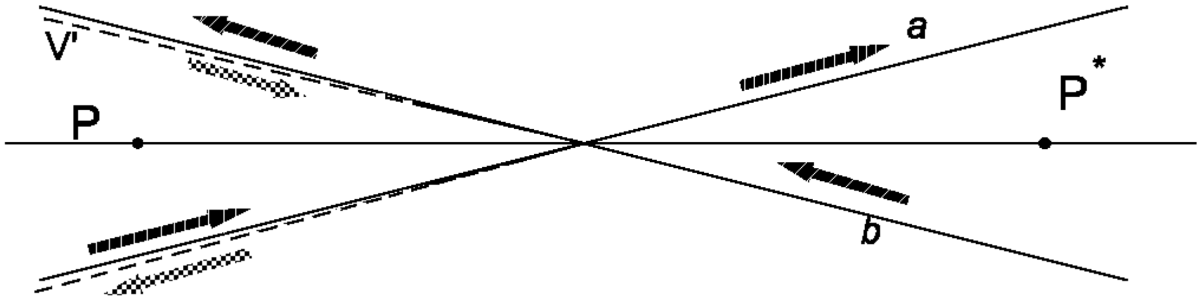
For symmetry reasons, the total field is twice that generated by each leg and has still the same direction.

2A. When $\alpha = \pi/2$ the "V" becomes a straight infinite wire. In this case the magnitude of the field B(P) is known to be $B = \dfrac{i}{2\pi\varepsilon_0 c^2 d} = \dfrac{i\mu_0}{2\pi d}$, and since $\tan(\pi/4)=1$, the factor k is $\dfrac{i\,\mu_0}{2\pi d}$.

The following solution is equally acceptable:

2B. If the student is aware of the equation $B = \dfrac{\mu_0 i}{4\pi}\dfrac{\cos\theta_1 - \cos\theta_2}{h}$ for a finite stretch of wire lying on a straight line at a distance h from point P and whose ends are seen from P under the angles θ_1 and θ_2, he can find that the two legs of the "V" both produce fields $\dfrac{\mu_0 i}{4\pi}\dfrac{1-\cos\alpha}{d\,\sin\alpha}$ and therefore the total field is $B = \dfrac{i\mu_0}{2\pi\, d}\dfrac{1-\cos\alpha}{\sin\alpha} = \dfrac{i\mu_0}{2\pi\, d}\tan\left(\dfrac{\alpha}{2}\right)$. This is a more complete solution since it also proves the angular dependence, but it is not required.

3A. In order to compute $\mathbf{B}(P^*)$ we may consider the "V" as equivalent to two crossed infinite wires (*a* and *b* in the following figure) plus another "V", symmetrical to the first one, shown in the figure as V', carrying the same current *i*, in opposite direction.



Then $B(\mathrm{P}^*) = B_a(\mathrm{P}^*) + B_b(\mathrm{P}^*) + B_{\mathrm{V'}}(\mathrm{P}^*)$. The individual contributions are:

$B_a(\mathrm{P}^*) = B_b(\mathrm{P}^*) = \dfrac{i\mu_0}{2\pi d \sin\alpha}$, along the negative z axis;

$B_{\mathrm{V'}}(\mathrm{P}^*) = \dfrac{i\mu_0}{2\pi d} \tan\left(\dfrac{\alpha}{2}\right)$, along the positive z axis.

Therefore we have $B(\mathrm{P}^*) = \dfrac{i\mu_0}{2\pi d}\left[\dfrac{2}{\sin\alpha} - \tan\left(\dfrac{\alpha}{2}\right)\right] = k\left(\dfrac{1+\cos\alpha}{\sin\alpha}\right) = k\cot\left(\dfrac{\alpha}{2}\right)$, and the field is along the negative z axis.

The following solutions are equally acceptable:

3B. The point P^* inside a "V" with half-span α can be treated as if it would be on the outside of a "V" with half-span π-α carrying the same current but in an opposite way, therefore the field is $B(\mathrm{P}^*) = k\tan\left(\dfrac{\pi-\alpha}{2}\right) = k\tan\left(\dfrac{\pi}{2} - \dfrac{\alpha}{2}\right) = k\cot\left(\dfrac{\alpha}{2}\right)$; the direction is still that of the z axis but it is along the negative axis because the current flows in the opposite way as previously discussed.

3C. If the student follows the procedure outlined under 2B., he/she may also find the field value in P* by the same method.

4. The mechanical moment **M** acting on the magnetic needle placed in point P is given by **M** = μ ∧ **B** (where the symbol ∧ is used for vector product). If the needle is displaced from its equilibrium position by an angle β small enough to approximate sinβ with β, the angular momentum theorem gives $M = -\mu B\beta = \dfrac{dL}{dt} = I\dfrac{d^2\beta}{dt^2}$, where there is a minus sign because the mechanical momentum is always opposite to the displacement from equilibrium. The period T of the small oscillations is therefore given by $T = \dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{I}{\mu B}}$.

Writing the differential equation, however, is not required: the student should recognise the same situation as with a harmonic oscillator.

5. If we label with subscript A the computations based on Ampère's interpretation, and with subscript BS those based on the other hypothesis by Biot and Savart, we have

$$B_{\mathrm{A}} = \frac{i\mu_0}{2\pi d}\tan\left(\frac{\alpha}{2}\right) \qquad B_{\mathrm{BS}} = \frac{i\mu_0}{\pi^2 d}\alpha$$

$$T_{\mathrm{A}} = 2\pi\sqrt{\frac{2\pi Id}{\mu_0 \mu i \tan\left(\frac{\alpha}{2}\right)}} \qquad T_{\mathrm{BS}} = 2\pi\sqrt{\frac{\pi^2 Id}{\mu_0 \mu i \alpha}}$$

$$\frac{T_{\mathrm{A}}}{T_{\mathrm{BS}}} = \sqrt{\frac{2\alpha}{\pi \tan\left(\frac{\alpha}{2}\right)}}$$

For $\alpha = \pi/2$ (maximum possible value) $T_{\mathrm{A}} = T_{\mathrm{BS}}$; and for $\alpha \to 0$ $T_{\mathrm{A}} \to \frac{2}{\sqrt{\pi}} T_{\mathrm{BS}} \approx 1.128\, T_{\mathrm{BS}}$. Since within this range $\frac{\tan(\alpha/2)}{\alpha/2}$ is a monotonically growing function of α, $\frac{T_{\mathrm{A}}}{T_{\mathrm{BS}}}$ is a monotonically decreasing function of α; in an experiment it is therefore not possible to distinguish between the two interpretations if the value of α is larger than the value for which $T_{\mathrm{A}} = 1.10\, T_{\mathrm{BS}}$ (10% difference), namely when $\tan\left(\frac{\alpha}{2}\right) = \frac{4}{1.21\pi}\frac{\alpha}{2} = 1.05\frac{\alpha}{2}$. By looking into the trigonometry tables or using a calculator we see that this condition is well approximated when $\alpha/2 = 0.38$ rad; α must therefore be smaller than 0.77 rad $\approx 44°$.

A graphical solution of the equation for α is acceptable but somewhat lengthy. A series development, on the contrary, is not acceptable.

Grading guidelines

1. 1 for recognising that each leg gives the same contribution
0.5 for a correct sketch

2. 0.5 for recognising that $\alpha=\pi/2$ for a straight wire, or for knowledge of the equation given in 2B.
0.25 for correct field equation (infinite or finite)
0.25 for value of k

3. 0.7 for recognising that the V is equivalent to two infinite wires plus another V
0.3 for correct field equation for an infinite wire
0.5 for correct result for the intensity of the required field
0.5 for correct field direction

alternatively

0.8 for describing the point as outside a V with π-α half-amplitude and opposite current
0.7 for correct analytic result
0.5 for correct field direction

alternatively

0.5 for correctly using equation under 2B
1 for correct analytic result
0.5 for correct field direction

4. 0.5 for correct equation for mechanical moment **M**
0.5 for doing small angle approximation $\sin \beta \approx \beta$
1 for correct equation of motion, including sign, or for recognizing analogy with harmonic oscillator
0.5 for correct analytic result for T

5. 0.3 for correct formulas of the two periods
0.3 for recognising the limiting values for α
0.4 for correct ratio between the periods
1 for finding the relationship between α and tangent
0.5 for suitable approximate value of α
0.5 for final explicit limiting value of α

For the numerical values a full score cannot be given if the number of digits is incorrect (more than one digit more or less than those given in the solution) or if the units are incorrect or missing

Solution

1A. Assuming – as outlined in the text – that the orbit is circular, and relating the radial acceleration $\frac{V^2}{R}$ to the gravitational field $\frac{GM_S}{R^2}$ (where M_S is the solar mass) we obtain Jupiter's orbital speed $V = \sqrt{\frac{GM_S}{R}} \approx 1.306 \cdot 10^4$ m/s.

The following alternative solution is also acceptable:

1B. Since we treat Jupiter's motion as circular and uniform, $V = \omega R = \frac{2\pi R}{y_J}$, where y_J is the revolution period of Jupiter, which is given in the list of the general physical constants.

2. The two gravitational forces on the space probe are equal when

$$\frac{GMm}{\rho^2} = \frac{GM_S m}{(R-\rho)^2} \qquad (2)$$

(where ρ is the distance from Jupiter and M is Jupiter's mass), whence

$$\sqrt{M}\,(R-\rho) = \rho\sqrt{M_S} \qquad (3)$$

and

$$\rho = \frac{\sqrt{M}}{\sqrt{M_S}+\sqrt{M}} R = 0.02997R = 2.333 \cdot 10^{10}\,\text{m} \qquad (4)$$

and therefore the two gravitational attractions are equal at a distance of about 23.3 million kilometers from Jupiter (about 334 Jupiter radii).

3. With a simple Galilean transformation we find that the velocity components of the probe in Jupiter's reference frame are

$$\begin{cases} v'_x = V \\ v'_y = v_0 \end{cases}$$

and therefore - in Jupiter's reference frame – the probe travels with an angle $\theta_0 = \arctan\frac{v_0}{V}$ with respect to the x axis and its speed is $v' = \sqrt{v_0^2 + V^2}$ (we also note that $\cos\theta_0 = \frac{V}{\sqrt{v_0^2+V^2}} = \frac{V}{v'}$

and $\sin\theta_0 = \dfrac{v_0}{\sqrt{v_0^2 + V^2}} = \dfrac{v_0}{v'}$).

Using the given values we obtain θ_0 = 0.653 rad ≈ 37.4° and v'=1.65·10⁴ m/s .

4. Since the probe trajectory can be described only approximately as the result of a two-body gravitational interaction (we should also take into account the interaction with the Sun and other planets) we assume a large but not infinite distance from Jupiter and we approximate the total energy in Jupiter's reference frame as the probe's kinetic energy at that distance:

$$E \approx \frac{1}{2} m v'^2 \tag{5}$$

The corresponding numerical value is E = 112 GJ.

5. Equation (1) shows that the radial distance becomes infinite, and its reciprocal equals zero, when

$$1 + \sqrt{1 + \frac{2Ev'^2 b^2}{G^2 M^2 m}} \cos\theta = 0 \tag{7}$$

namely when

$$\cos\theta = -\frac{1}{\sqrt{1 + \frac{2Ev'^2 b^2}{G^2 M^2 m}}} \tag{8}$$

We should also note that the radial distance can't be negative, and therefore its acceptable values are those satisfying the equation

$$1 + \sqrt{1 + \frac{2Ev'^2 b^2}{G^2 M^2 m}} \cos\theta \geq 0 \tag{9}$$

or

$$\cos\theta \geq -\frac{1}{\sqrt{1 + \frac{2Ev'^2 b^2}{G^2 M^2 m}}} \tag{10}$$

The solutions for the limiting case of eq. (10) (i.e. when the equal sign applies) are:

$$\theta_{\pm} = \pm \arccos\left[-\left(1 + \frac{2Ev'^2 b^2}{G^2 M^2 m}\right)^{-1/2} \right] = \pm\left(\pi - \arccos\frac{1}{\sqrt{1 + \frac{2Ev'^2 b^2}{G^2 M^2 m}}} \right) \tag{11}$$

and therefore the angle $\Delta\theta$ (shown in figure 2) between the two hyperbola asymptotes is given by:

$$\begin{aligned}\Delta\theta &= (\theta_+ - \theta_-) - \pi \\ &= \pi - 2\arccos\frac{1}{\sqrt{1+\dfrac{2Ev'^2 b^2}{G^2M^2m}}} \qquad (12) \\ &= \pi - 2\arccos\frac{1}{\sqrt{1+\dfrac{v'^4 b^2}{G^2M^2}}}\end{aligned}$$

In the last line, we used the value of the total energy as computed in the previous section.

6. The angular deviation is a monotonically decreasing function of the impact parameter, whence the deviation has a maximum when the impact parameter has a minimum. From the discussion in the previous section we easily see that the point of nearest approach is when $\theta = 0$, and in this case the minimum distance between probe and planet center is easily obtained from eq. (1):

$$r_{\min} = \frac{v'^2 b^2}{GM}\left(1+\sqrt{1+\frac{v'^4 b^2}{G^2M^2}}\right)^{-1} \qquad (13)$$

By inverting equation (13) we obtain the impact parameter

$$b = \sqrt{r_{\min}^2 + \frac{2GM}{v'^2} r_{\min}} \qquad (14)$$

We may note that this result can alternatively be obtained by considering that, due to the conservation of angular momentum, we have

$$L = mv'b = mv'_{\min}\, r_{\min}$$

where we introduced the speed corresponding to the nearest approach. In addition, the conservation of energy gives

$$E = \frac{1}{2}mv'^2 = \frac{1}{2}mv'^2_{\min} - \frac{GMm}{r_{\min}}$$

and by combining these two equations we obtain equation (14) again.

The impact parameter is an increasing function of the distance of nearest approach; therefore, if the probe cannot approach Jupiter's surface by less than two radii (and thus $r_{\min} = 3R_B$, where R_B is Jupiter's body radius), the minimum acceptable value of the impact parameter is

$$b_{\min} = \sqrt{9R_B^2 + \frac{6GM}{v'^2} R_B} \qquad (15)$$

From this equation we finally obtain the maximum possible deviation:

$$\Delta\theta_{\max} = \pi - 2\arccos\frac{1}{\sqrt{1+\frac{v'^4 b_{\min}^2}{G^2M^2}}} = \pi - 2\arccos\frac{1}{\sqrt{1+\frac{v'^4}{G^2M^2}\left(9R_B^2 + \frac{6GM}{v'^2}R_B\right)}} \qquad (16)$$

and by using the numerical values we computed before we obtain:

$b_{\min} = 4.90 \cdot 10^8$ m $\approx 7.0\ R_B$ and $\Delta\theta_{\max} = 1.526$ rad $\approx 87.4°$

7. The final direction of motion with respect to the x axis in Jupiter's reference frame is given by the initial angle plus the deviation angle, thus $\theta_0 + \Delta\theta$ if the probe passes behind the planet. The final velocity components in Jupiter's reference frame are therefore:

$$\begin{cases} v'_x = v' \cos(\theta_0 + \Delta\theta) \\ v'_y = v' \sin(\theta_0 + \Delta\theta) \end{cases}$$

whereas in the Sun reference frame they are

$$\begin{cases} v''_x = v' \cos(\theta_0 + \Delta\theta) - V \\ v''_y = v' \sin(\theta_0 + \Delta\theta) \end{cases}$$

Therefore the final probe speed in the Sun reference frame is

$$
\begin{aligned}
v'' &= \sqrt{(v'\cos(\theta_0+\Delta\theta)-V)^2+(v'\sin(\theta_0+\Delta\theta))^2} \\
&= \sqrt{v_0^2+2V^2-2v'V\cos(\theta_0+\Delta\theta)} \\
&= \sqrt{v_0^2+2V^2-2v'V\left(\cos\theta_0\cos\Delta\theta-\sin\theta_0\sin\Delta\theta\right)} \qquad (17)\\
&= \sqrt{v_0^2+2V^2-2V\left(V\cos\Delta\theta-v_0\sin\Delta\theta\right)} \\
&= \sqrt{v_0\left(v_0+2V\sin\Delta\theta\right)+2V^2\left(1-\cos\Delta\theta\right)}
\end{aligned}
$$

8. Using the value of the maximum possible angular deviation, the numerical result is $v'' = 2.62\cdot 10^4$ m/s.

Grading guidelines

1.	0.4	Law of gravitation, or law of circular uniform motion
	0.4	Correct approach
	0.4+0.3	Correct results for velocity of Jupiter
2.	0.3	Correct approach
	0.4+0.3	Correct results for distance from Jupiter
3.	1	Correct transformation between reference frames
	0.3+0.2	Correct results for probe speed in Jupiter reference frame
	0.3+0.2	Correct results for probe angle
4.	0.8	Understanding how to handle the potential energy at infinity
	0.2	Numerical result for kinetic energy
5.	0.6	Correct approach
	0.6	Equation for the orientation of the asymptotes
	0.8	Equation for the probe deflection angle
6.	0.3+0.2	Correct results for minimum impact parameter
	0.3+0.2	Correct results for maximum deflection angle
7.	0.5	Equation for velocity components in the Sun reference frame
	0.5	Equation for speed as a function of angular deflection
8.	0.5	Numerical result for final speed

For “correct results” two possible marks are given: the first one is for the analytical equation and the second one for the numerical value.
For the numerical values a full score cannot be given if the number of digits is incorrect (more than one digit more or less than those given in the solution) or if the units are incorrect or missing.